

On the Possibility of Measuring the Abraham Force using Whispering Gallery Modes

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Critical experimental tests of the time-dependent Abraham force in phenomenological electrodynamics are scarce. In this paper we analyze the possibility of making use of intensity-modulated whispering gallery modes in a microresonator for this purpose. Systems of this kind appear attractive, as the strong concentration of electromagnetic fields near the rim of the resonator serves to enhance the Abraham torque exerted by the field. We analyze mainly spherical resonators, although as an introductory step we consider also the cylinder geometry. The order of magnitude of the Abraham torques are estimated by inserting reasonable and common values for the various input parameters. As expected, the predicted torques turn out to be very small, although probably not beyond any reach experimentally. Our main idea is essentially a generalization of the method used by G. B. Walker *et al.* [Can. J. Phys. **53**, 2577 (1975)] for low-frequency fields, to the optical case.

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I. INTRODUCTION

The one-hundred years old Abraham-Minkowski energy-momentum problem in phenomenological electrodynamics [1, 2] has recently attracted considerable interest. Assume henceforth for simplicity that the medium is nonmagnetic and nondispersive, with refractive index n . In our opinion – as expressed in the review article some years ago by one of the present authors [3] – the most physical expression for the electromagnetic force density is the Abraham expression (SI units assumed)

$$\mathbf{f}^A = \mathbf{f}^{AM} + \frac{n^2 - 1}{c^2} \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{H}). \quad (1.1)$$

Here the first term $\mathbf{f}^{AM} = -(\epsilon_0/2)E^2\nabla n^2$ is different from zero in regions where n varies with position, especially in the surface regions of dielectrics. This term is common for the Abraham and Minkowski tensors, and may appropriately be called the Abraham-Minkowski term. The second, time-dependent term in Eq. (1.1), is the Abraham term. It may be noted that the expression (1.1) is in agreement with Ginzburg [4], as well as with Landau and Lifshitz [5].

One may ask: is it possible to detect the Abraham term in experiment? The answer is yes, but the task has proven to be surprisingly difficult. The magnitude of the electromagnetic frequency is a significant factor in this context. Let us give a brief account of three important experimental cases:

1) The first case is the quasi-stationary torque experiment of Walker *et al.* [6, 7]. Strong, time-varying, orthogonal electric and magnetic fields were applied across a dielectric shell of high permittivity, making it possible to detect the *oscillations themselves*. In this way the Abraham term was measured quantitatively.

2) When considering instead high-frequency fields such as in optics, the Abraham term fluctuates out when averaged over a period. One can thus no longer detect this force directly. The physical effect of this force is however to produce an accompanying mechanical momentum propagating together with the Abraham momentum. The resulting total momentum is the Minkowski momentum, corresponding to the divergence-free Minkowski energy-momentum tensor. This tensor has the particular property of being space-like, corresponding to the possibility of getting negative field energy in certain inertial frames. An authoritative experiment measuring the Minkowski momentum is that of Jones *et al.* [8, 9], measuring the radiation pressure on a mirror immersed in a dielectric liquid. Both cases 1) and 2) are discussed in some detail in Ref. [3].

3) The third example to be mentioned is the photon recoil experiment of Campbell *et al.* [10], where the photon momentum in a medium (in this case a Bose-Einstein condensate) was found to be equal to the Minkowski value $\hbar\mathbf{k}$.

Most other experiments are measuring not the Abraham term but rather the surface force \mathbf{f}^{AM} , although claims are sometimes made to the contrary. In our opinion this is the case also for the interesting new fiber optical experiment of She *et al.* [11]; cf. the remarks in Refs. [12, 13].

Our main purpose in the present paper is however not to interpret already existing experiments, but instead to propose the idea of using *whispering gallery modes* as a convenient experimental tool to detect the Abraham term in optics. To our knowledge this idea has not been considered before. Whispering gallery modes are commonly produced in microspheres; they have a large circulating power, about 100 W typically, and the field energy is concentrated along the rim of the sphere. That means, if such a sphere is suspended in the gravitational field and fed with an appropriate intensity modulated field, the sphere becomes exposed to a vertical torque according to Eq. (1.1). With the field energy essentially concen-

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trated along the rim, the arm in the torque calculation is essentially the same as the radius, thus maximizing the torque. In effect, this is the idea of the experiment of Walker *et al.* [6, 6], generalized to optical frequencies. We have actually suggested this idea qualitatively before, in Refs. [12, 13].

The next two sections give quantitative estimates for performing such an experiment. The torque turns out to be small, as expected, but not beyond any possibility for experimental detection. Spherical geometry, as mentioned, is most typical for the whispering gallery setup. In the next section we consider however as an introductory step the somewhat more simple geometry of a cylindrical shell.

Before closing this section, let us give a few more references to the Abraham-Minkowski problem, in addition to the references given above. A nice introduction can be found in Møller's book [14]. A review, up to 2007, is given by Pfeifer *et al.* [15]. Some more recent papers are Refs. [16–18].

II. CYLINDRICAL GEOMETRY

Consider first as the simplest case a compact cylinder of length L and radius a . On the inside, $r < a$, the permittivity is ϵ and the permeability μ . On the outside, $r > a$, a vacuum is assumed. The dispersion relation for stationary modes is known to be [19]

$$\left[\frac{\mu J'_m(u)}{u J_m(u)} - \frac{1}{v} \frac{H_m^{(1)'}(v)}{H_m^{(1)}(v)} \right] \left[\frac{\epsilon \omega^2 J'_m(u)}{u J_m(u)} - \frac{\omega^2 H_m^{(1)'}(v)}{v H_m^{(1)}(v)} \right] = m^2 k^2 \left(\frac{1}{v^2} - \frac{1}{u^2} \right)^2. \quad (2.1)$$

We are working with SI units and let ϵ and μ be dimensional, so that $\mathbf{D} = \epsilon \mathbf{E}$, $\mathbf{B} = \mu \mathbf{H}$. The transverse wave vectors on the inside and the outside are

$$\lambda_1 = n\omega/c, \quad \lambda_2 = \omega/c, \quad (2.2)$$

respectively, while their nondimensional counterparts are

$$u = \lambda_1 a, \quad v = \lambda_2 a. \quad (2.3)$$

An important property of this equation is that when the axial wave vector $k = 0$ – as is of interest here as we consider azimuthal modes only – the right-hand side vanishes and the problem becomes separable into TE and TM modes.

We write the mode expansions for the fields in the inner

region [19]:

$$E_r = -\frac{\mu\omega}{\lambda_1^2 r} \sum_{m=-\infty}^{\infty} m J_m(\lambda_1 r) b_m F_m, \quad (2.4a)$$

$$E_\theta = -\frac{i\mu\omega}{\lambda_1} \sum_{m=-\infty}^{\infty} J'_m(\lambda_1 r) b_m F_m, \quad (2.4b)$$

$$E_z = \sum_{m=-\infty}^{\infty} J_m(\lambda_1 r) a_m F_m, \quad (2.4c)$$

and

$$H_r = \frac{\epsilon\omega}{\lambda_1^2 r} \sum_{m=-\infty}^{\infty} m J_m(\lambda_1 r) a_m F_m, \quad (2.5a)$$

$$H_\theta = \frac{i\epsilon\omega}{\lambda_1} \sum_{m=-\infty}^{\infty} J'_m(\lambda_1 r) a_m F_m, \quad (2.5b)$$

$$H_z = \sum_{m=-\infty}^{\infty} J_m(\lambda_1 r) b_m F_m, \quad (2.5c)$$

where

$$F_m = e^{im\theta - i\omega t}. \quad (2.6)$$

The coefficients a_m and b_m , corresponding to the TM and TE modes, give the weight of each mode.

In our considerations below we will for simplicity extract one single TE mode of high order m , such that there is an azimuthally moving momentum concentrated in the vicinity of the boundary $r = a$. (In reality, the incident power may be distributed over a band of neighbouring m modes, but this does not influence the essence of our argument.) We first need to determine the magnitude of the radial argument $\lambda_1 r \approx u$. Let us take

$$m = 100, \quad n = 1.5, \quad a = 100 \text{ } \mu\text{m}. \quad (2.7)$$

It is known that for a large value of the order m the first maximum of the function $J_m(x)$ occurs when x is very close to m . This maximum is the one of interest here. Thus the lowest resonance frequency ω is determined by the equation

$$na\omega/c = m. \quad (2.8)$$

With the numbers given above,

$$\omega = 2 \times 10^{16} \text{ s}^{-1}. \quad (2.9)$$

In this manner we manage to make the beam strongly concentrated near the rim, as desired. One has in this case $E_z = 0$, $H_r = 0$, while the nonvanishing field components of interest are

$$E_r = -\frac{\mu\omega}{\lambda_1^2 r} m J_m(\lambda_1 r) b_m F_m, \quad (2.10)$$

$$H_z = J_m(\lambda_1 r) b_m F_m. \quad (2.11)$$

The azimuthal component of the Poynting vector $\mathbf{S}(r)$ in the interior is

$$S_\theta(r) = -\frac{1}{2}\Re[E_r H_z^*] = \frac{\mu\omega m}{2\lambda_1^2 r} J_m^2(\lambda_1 r) |b_m|^2, \quad (2.12)$$

corresponding to the azimuthal power

$$P = L \int_0^a S_\theta dr = \frac{\mu\omega m L}{2\lambda_1^2} |b_m|^2 \int_0^u \frac{dx}{x} J_m^2(x). \quad (2.13)$$

In our case the factor $1/x$ can be extracted outside the integral, so that

$$P = \frac{\mu\omega m L}{2\lambda_1^2} |b_m|^2 \int_0^u dx J_m^2(x). \quad (2.14)$$

Assume now that the beam is intensity modulated with a frequency ω_0 (ω_0 low compared with optical frequencies),

$$P = P_0 \cos \omega_0 t, \quad S_\theta = S_0 \cos \omega_0 t. \quad (2.15)$$

Then the azimuthal Abraham force density f_ϕ^A is

$$f_\phi^A = \frac{n^2 - 1}{c^2} \frac{\partial S_\theta}{\partial t} = -\frac{n^2 - 1}{c^2} \omega_0 S_0 \sin \omega_0 t, \quad (2.16)$$

giving rise to the following Abraham torque N_z^A around the vertical symmetry axis:

$$N_z^A = 2\pi L \int_0^a r^2 f_\phi^A dr \approx 2\pi L a^2 \int_0^a f_\phi^A dr. \quad (2.17)$$

Defining the quantity K as

$$K = -\frac{n^2 - 1}{c^2} 2\pi a^2 P_0, \quad (2.18)$$

we thus see that the torque can be written as

$$N_z^A = K \omega_0 \sin \omega_0 t. \quad (2.19)$$

As expected, the torque becomes very small. As order of magnitude we get

$$K \sim \frac{2\pi a^2}{c^2} P_0 \sim (0.7 \times 10^{-24} \text{s}^2) \cdot P_0 \quad (2.20)$$

and the Abraham torque is estimated as

$$N_z^A \sim (0.7 \times 10^{-24} \text{s}^2) \cdot \omega_0 P_0. \quad (2.21)$$

Insert first the very low value of $\omega_0 \sim 1 \text{ s}^{-1}$, and take $P_0 \sim 100 \text{ W}$. We get $N_z \sim 0.7 \times 10^{-22} \text{ N m}$, which is much less than the value 10^{-16} N m obtained in the classic Beth experiment [20], for example, in which the angular momentum of light was measured. It is however possible to improve the situation by exploiting the fact that the build-up and ringdown times for this kind of resonators are known to be very small, in the order of tens to hundreds of ns (see discussion below). It is thus realistic to insert a much higher value for ω_0 . Inserting

tentatively $\omega_0 = 1000 \text{ s}^{-1}$ we get $N_z \sim 0.7 \times 10^{-19} \text{ N m}$, which is perhaps not so unrealistic after all.

It is physically instructive to look at the system in another way, by considering the angular deflection ϕ of the cylinder instead of the magnitude of the torque. Let the cylinder be hanging vertically in the gravitational field, suspended by a thin wire of known torsion constant κ . Denoting the eigenfrequency of the cylinder in the absence of any torque by Ω , and denoting the damping coefficient by γ , we have as equation of motion

$$\ddot{\phi} + \gamma \dot{\phi} + \Omega^2 \phi = \frac{K}{I} \omega_0 \sin \omega_0 t. \quad (2.22)$$

Here $I = \frac{1}{2} M a^2$ is the moment of inertia about the z axis, $M = \rho a^2 L$ being the cylinder mass with ρ the material density. In our notation, $\kappa = I \Omega^2$. With $a = 100 \mu\text{m}$ as above we obtain, when choosing $L = 1 \text{ mm}$ and assuming $\rho \sim 10^3 \text{ kg/m}^3$,

$$\Omega = \sqrt{\kappa/I} \sim 10^8 \sqrt{\kappa}. \quad (2.23)$$

For the magnitude of κ we may choose a typical value characteristic for torsion experiments testing the equivalence principle, $\kappa \sim 10^{-9} \text{ N m/rad}$ [21, 22]. Then,

$$\Omega \sim 10^3 \text{ rad s}^{-1}. \quad (2.24)$$

The magnitude of Ω is large because a is assumed small.

The largest oscillations occur at resonance, when ω_0 is chosen equal to Ω . Then,

$$\phi = -\frac{K}{I\gamma} \cos \Omega t. \quad (2.25)$$

The maximum value, when $P_0 \sim 100 \text{ W}$, is

$$\phi_{\max} = \frac{n^2 - 1}{c^2} \frac{4}{M} \frac{P_0}{\gamma} \sim \frac{10^{-7} \text{ rad}}{\gamma \text{ s}}. \quad (2.26)$$

It would be of interest to make an estimate of the damping constant γ here, but we postpone that until the next section.

Notice that the very existence of an oscillatory movement would be enough to make the experiment critical with respect to the Abraham force. The Minkowski tensor does not predict there to be an azimuthal movement at all.

III. SPHERICAL GEOMETRY

As mentioned above, whispering gallery modes are usually associated with microspheres. Let the radius of the sphere be denoted by a . As above, we look for the eigenmodes, and we will for simplicity focus on the TE modes only. (The meaning of the symbol TE is here that the electric field is transverse to the radius vector \mathbf{r} .) We introduce quantities α and \tilde{r} defined by

$$\alpha = \omega a / c, \quad \tilde{r} = r / a. \quad (3.1)$$

Thus α is the magnitude of the nondimensional wave vector in the exterior region (vacuum), whereas $\tilde{r} = 1$ at the boundary. Making use of the Riccati-Bessel function

$$\psi_l(x) = x j_l(x), \quad (3.2)$$

the basic TE modes in the interior can conveniently be written as

$$E_r = 0, \quad (3.3a)$$

$$E_\theta = -\frac{imA_{lm}}{n\alpha\tilde{r}} \frac{P_l^m(\cos\theta)}{\sin\theta} \psi_l(n\alpha\tilde{r}) F_m, \quad (3.3b)$$

$$E_\phi = \frac{A_{lm}}{n\alpha\tilde{r}} \frac{dP_l^m(\cos\theta)}{d\theta} \psi_l(n\alpha\tilde{r}) F_m, \quad (3.3c)$$

and

$$H_r = -\frac{l(l+1)}{i\omega\mu} \frac{A_{lm}}{n\alpha\tilde{r}^2} \frac{1}{a} P_l^m(\cos\theta) \psi_l(n\alpha\tilde{r}) F_m, \quad (3.4a)$$

$$H_\theta = -\frac{1}{i\omega\mu} \frac{A_{lm}}{\tilde{r}} \frac{1}{a} \frac{dP_l^m(\cos\theta)}{d\theta} \psi'_l(n\alpha\tilde{r}) F_m, \quad (3.4b)$$

$$H_\phi = -\frac{m}{\omega\mu\sin\theta} \frac{A_{lm}}{\tilde{r}} \frac{1}{a} P_l^m(\cos\theta) \psi'_l(n\alpha\tilde{r}) F_m, \quad (3.4c)$$

where A_{lm} are constants, and

$$F_m = e^{im\phi - i\omega t}. \quad (3.5)$$

The mode expansions above essentially follow Stratton [19].

The components of Poynting's vector are, when averaged over an optical period,

$$S_r = \frac{1}{2} \Re[E_\theta H_\phi^* - E_\phi H_\theta^*], \quad (3.6a)$$

$$S_\theta = \frac{1}{2} \Re[E_\phi H_r^*], \quad (3.6b)$$

$$S_\phi = -\frac{1}{2} \Re[E_\theta H_r^*]. \quad (3.6c)$$

Assume that the sphere is fed by an incident flux from the outside such that only the component S_ϕ of \mathbf{S} in the interior is different from zero. With an intensity modulated energy flux such as above, $S_\phi = S_0 \cos \omega_0 t$, we thus get for the azimuthally directed Abraham force density in the interior

$$f_\phi^A = -\frac{n^2 - 1}{c^2} \omega_0 S_0 \sin \omega_0 t. \quad (3.7)$$

From the above expressions,

$$S_0 = \frac{m}{2(n\alpha)^2 \tilde{r}^3} \frac{l(l+1)}{\omega\mu} \frac{|A_{lm}|^2}{a} \frac{[P_l^m]^2}{\sin\theta} \psi_l^2. \quad (3.8)$$

The Abraham torque, directed along the z axis, then becomes

$$N_z^A = \int (\mathbf{r} \times \mathbf{f}^A)_z dV = \int r f_\phi^A \sin\theta dV, \quad (3.9)$$

where the integration is over the sphere, with $dV = r^2 \sin\theta dr d\theta d\phi$. Making use of Eqs. (3.7) and (3.8) we obtain

$$N_z^A = -\frac{n^2 - 1}{c^2} \frac{\pi m a^3}{(n\alpha)^2} \frac{l(l+1)}{\omega\mu} |A_{lm}|^2 \omega_0 K_I K_{II} \sin \omega_0 t, \quad (3.10)$$

where K_I and K_{II} are the integrals

$$K_I = \int_0^1 \psi_l^2(n\alpha\tilde{r}) d\tilde{r} = \frac{1}{2} [\psi_l^2(n\alpha) - \psi_{l-1}(n\alpha)\psi_{l+1}(n\alpha)], \quad (3.11a)$$

$$K_{II} = \int_0^\pi [P_l^m(\cos\theta)]^2 \sin\theta d\theta = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!}. \quad (3.11b)$$

We want to relate this to the total power P flowing in the azimuthal direction in the sphere. We calculate P by integrating S_ϕ over the area of a semicircle with radius a ,

$$P = \int_0^\pi d\theta \int_0^a r dr S_\phi = \frac{ma}{2(n\alpha)^2} \frac{l(l+1)}{\omega\mu} |A_{lm}|^2 K_{III} K_{IV} \cos \omega_0 t, \quad (3.12)$$

where

$$K_{III} = \int_0^1 \frac{d\tilde{r}}{\tilde{r}^2} \psi_l^2(n\alpha\tilde{r}), \quad (3.13a)$$

$$K_{IV} = \int_0^\pi \frac{[P_l^m(\cos\theta)]^2}{\sin\theta} d\theta. \quad (3.13b)$$

As before, it is assumed that the supplied power is intensity modulated, $P = P_0 \cos \omega_0 t$.

The two last integrals can be processed further, at least approximatively. First, we can rewrite K_{III} as

$$K_{III} = \frac{1}{2} \pi n\alpha \int_0^{n\alpha} \frac{dx}{x} J_\nu^2(x), \quad (3.14)$$

with $\nu = l + 1/2$. For actual physical values, $n\alpha \gg 1$. We can thus replace the upper limit with infinity, and make use of formula 6.574.2 in Ref. [23] to get

$$K_{III} \approx \frac{\pi n\alpha}{2(2l+1)}. \quad (3.15)$$

Finally, the integral K_{IV} is simply (cf. formula 8.14.14 in Ref. [24])

$$K_{IV} = \frac{(l+m)!}{m(l-m)!}. \quad (3.16)$$

We are now able to relate the torque N_z^A to the power P . The result becomes quite simple:

$$N_z^A = -\frac{n^2 - 1}{c^2} \frac{4ma^2\omega_0}{n\alpha} P_0 \sin \omega_0 t \times [\psi_l^2(n\alpha) - \psi_{l-1}(n\alpha)\psi_{l+1}(n\alpha)]. \quad (3.17)$$

The radius of the sphere is seen to appear in the prefactor a^2 , as well as in the nondimensional parameter $\alpha = \omega a/c$. The parameter l occurs only as an order parameter in the function ψ_l . We see that the torque is proportional to m . This is as we would expect, as the whispering gallery modes are associated with $m = l$, i.e. the maximum value of m . It should correspond to a maximum angular momentum and accordingly a maximum torque.

To proceed quantitatively, the value of α has to be determined. For the TE modes it is determined by the dispersion relation [19]

$$\frac{n\mu_0}{\mu} \frac{\psi_l'(n\alpha)}{\psi_l(n\alpha)} = \frac{\xi_l^{(1)'}(\alpha)}{\xi_l^{(1)}(\alpha)}, \quad (3.18)$$

where $\xi_l^{(1)}(x) = x h_l^{(1)}(x)$ is another member of the Riccati-Bessel functions. The equation (3.18) is complex and does not in general have real solutions, but approximate solutions with only a small imaginary inequality are found close to $\alpha \approx l$ for $l \gg 1$.

As at the end of the previous section, we focus now attention on the magnitude of the angular deflection ϕ , as this is most likely the quantity of main experimental interest. Without changing the notation we write the Abraham torque in the form $N_z^A = K\omega_0 \sin \omega_0 t$ as before, where now

$$K = -\frac{n^2 - 1}{c^2} \frac{4ma^2}{n\alpha} \times [\psi_l^2(n\alpha) - \psi_{l-1}(n\alpha)\psi_{l+1}(n\alpha)]P_0. \quad (3.19)$$

The equation of motion for ϕ takes the same form (2.22) as before, where now the moment of inertia is

$$I = \frac{2}{5}Ma^2 = \frac{8\pi}{15}\rho a^5, \quad (3.20)$$

M being the mass of the sphere. For definiteness let us take $a = 100 \mu\text{m}$. Then, with $\rho \sim 10^3 \text{ kg/m}^3$ we get $M \approx 4 \mu\text{g}$ and so, with $\kappa \sim 10^{-9} \text{ N m/rad}$ as before,

$$\Omega \sim 10^8 \sqrt{\kappa} \sim 10^3 \text{ rad s}^{-1}. \quad (3.21)$$

With these numerical choices the value of Ω becomes of the same order as in the cylinder case. The magnitude ϕ_{max} of the maximum deflection at resonance $\omega_0 = \Omega$ is now

$$\phi_{\text{max}} = \frac{10m}{Mn\alpha} \frac{n^2 - 1}{c^2} \times [\psi_l^2(n\alpha) - \psi_{l-1}(n\alpha)\psi_{l+1}(n\alpha)] \frac{P_0}{\gamma}. \quad (3.22)$$

As we have assumed $l \gg 1$ and $n\alpha \gg 1$ but otherwise left the ratio of these quantities unspecified, the ψ_l functions ought to be calculated numerically.

Let us finally make an estimate of the magnitude of the damping coefficient γ , assuming for definiteness that the damping is due to the viscosity of air only. We then

need to know the viscous torque on a sphere executing rotary oscillations about its symmetry axis. The solution of this problem is shown in Ref. [25]. An important parameter in this context is the penetration depth $\delta = \sqrt{2\nu/\Omega}$, where ν is the kinematic viscosity of the surrounding medium. For air, $\nu = 1.5 \times 10^{-5} \text{ m}^2/\text{s}$. Thus with $\Omega \sim 10^3 \text{ rad s}^{-1}$ we get $\delta \sim 170 \mu\text{m}$, which is of the same order as a . Strictly speaking we should therefore have to use the complete expression for the viscous torque, which is somewhat complicated. For our order-of-magnitude considerations it is however sufficient to use the simple expression

$$(N_z)_{\text{viscous}} \approx 8\pi\eta a^3 \Omega, \quad (3.23)$$

(corresponding mathematically to the $a/\delta \ll 1$ limit), where $\eta = 1.8 \times 10^{-5} \text{ Pa s}$ is the dynamic viscosity for air. Identifying $(N_z)_{\text{viscous}}$ with $I\gamma\Omega$ in accordance with Eq. (2.22), we get for the damping coefficient

$$\gamma = \frac{8\pi\eta}{I} a^3 \sim 30 \text{ s}^{-1}, \quad (3.24)$$

and the expression (3.22) for the maximum deflection can finally be written as

$$\phi_{\text{max}} = \frac{m}{2\pi n\alpha\eta a} \frac{n^2 - 1}{c^2} \times [\psi_l^2(n\alpha) - \psi_{l-1}(n\alpha)\psi_{l+1}(n\alpha)]P_0. \quad (3.25)$$

As expected, the deflection is very small. Whereas numerical evaluation of the ψ_l functions in general is called for, as mentioned, we may note that in cases where $l \ll n\alpha$ the approximation $\psi_l(n\alpha) \approx \sin(n\alpha - \frac{l\pi}{2})$ is useful. One can moreover obtain a simple estimate of the magnitude in the cylinder case by inserting γ from Eq. (3.24) into Eq. (2.26), whereby one finds $\phi_{\text{max}} \sim 10^{-8} \text{ rad}$. Careful adjustments of input parameters are obviously needed if the effect is to be verified experimentally.

IV. ON THE MAGNITUDE OF TORQUES IN EXISTING EXPERIMENTS

We close this investigation by making some estimates of radiation torques on spheres, as well as on ring resonators (a closely related geometry), for already existing experiments. As first example we take the setup reported in Ref. [26], where an infrared laser of wavelength $\lambda = 1500 \text{ nm}$ was used. Two different sphere radii were investigated, $a = 40 \mu\text{m}$ and $a = 70 \mu\text{m}$, corresponding to values of $\alpha \approx l = m$ equal to 162 and 283, respectively. Although the feeding laser had a power in the order of tens of microwatts to milliwatts, the extremely high Q factor of the silica sphere meant the buildup of circulating modes in the sphere grew enormous. Circulating powers in excess of 100W are routinely reported in such systems (e.g. [27]) (although this quantity was not explicitly given in the reference [26]). The refractive index

of materials used for ultra-high- Q spherical resonators, such as fused silica [26, 28] and quartz [29], are about $n = 1.5$. With these values as input for P_0 we obtain the torques $[N_z^A = N_0 \sin \omega_0 t]$

$$N_0 \approx \begin{cases} 4 \times 10^{-24} \text{Nms} \cdot \omega_0 \\ 1 \times 10^{-23} \text{Nms} \cdot \omega_0 \end{cases}, \text{ for } a = \begin{cases} 40 \mu\text{m} \\ 70 \mu\text{m} \end{cases}. \quad (4.1)$$

Note in general that for a sphere, $N_z^A \propto a$ according to Eq. (3.17), whereas $\phi_{\text{max}} \propto a^{-2}$ according to Eq. (3.25) when the viscous damping is accounted for.

The geometry of Ref. [27], which reports circulating powers in excess of 100 W, employs the toroidal ring resonator. This geometry has the benefit of having smaller mass and therefore smaller moment of inertia than a sphere of the same radius, allowing for larger angular deflections. For a thin ring the moment of inertia is

$$I_{\text{toroid}} \approx 2\pi\rho Aa^3 \quad (4.2)$$

where A is the area of cross-section. The torque on such a toroid would be roughly similar to that on a sphere, so it is reasonable to assume the angular deflection to be larger and scale as a^{-1} . This could allow larger radii

which could be beneficial for detection. Cf. also the review article [30].

We wish finally to re-emphasize the possibility of using quite high frequencies ω_0 in order to produce measurable values for the Abraham torque. We assumed above the strong field inside the microcavity to react instantaneously to the sinusoidal variations of the input signal, an approximation which is good provided the build-up and ringdown time (τ) of the resonator is small compared to $2\pi/\omega_0$. For the 45 μm radius toroidal resonator in Ref. [31], for example, a ringdown time of about 43 ns was measured. For cavities of even higher Q -factor, ringdown times are somewhat longer, yet this implies that we may choose tuning frequencies ω_0 as high as 10^6 without invalidating the theory. Due to the proportionality of the torque with ω_0 , going close to the megahertz regime could increase the torque to perhaps 10^{-17} Nm for a sphere with radius of some tens of microns.

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- [1] M. Abraham, *Rend. Circ. Matem. Palermo* **28**, 1 (1909); *ibid.* **30**, 33 (1910).
 - [2] H. Minkowski, *Nachr. Königl. Ges. Wiss. Göttingen* p. 53 (1908); reprinted as *Math. Ann.* **68**, 472 (1910).
 - [3] I. Brevik, *Phys. Reports* **52**, 133 (1979).
 - [4] V. L. Ginzburg, *Applications of Electrodynamics in Theoretical Physics and Astrophysics* (Gordon and Breach, New York, 1989).
 - [5] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, 2nd ed. (Butterworth-Heinemann, Oxford, 1984).
 - [6] G. B. Walker, D. G. Lahoz and G. Walker, *Can. J. Phys.* **53**, 2577 (1975).
 - [7] G. B. Walker and D. G. Lahoz, *Nature* **253**, 339 (1975).
 - [8] R. V. Jones and J. C. Richards, *Proc. R. Soc. A* **221**, 480 (1954).
 - [9] R. V. Jones and B. Leslie, *Proc. R. Soc. A* **360**, 347 (1978).
 - [10] G. K. Campbell, A. E. Leanhardt, J. Mun, M. Boyd, E. W. Streed, W. Ketterle, and D. E. Pritchard, *Phys. Rev. Lett.* **94**, 170403 (2005).
 - [11] W. She, J. Yu, and R. Feng, *Phys. Rev. Lett.* **101**, 243601 (2008).
 - [12] I. Brevik, *Phys. Rev. Lett.* **103**, 219301 (2009).
 - [13] I. Brevik and S. Å. Ellingsen, *Phys. Rev. A* **81**, 011806(R) (2010).
 - [14] C. Möller, *The Theory of Relativity*, 2nd. ed. (Clarendon Press, Oxford, 1972).
 - [15] R. N. C. Pfeifer, T. A. Nieminen, N. R. Heckenberg, and H. Rubinsztein-Dunlop, *Rev. Mod. Phys.* **79**, 1197 (2007).
 - [16] I. Brevik and S. A. Ellingsen, *Phys. Rev. A* **79**, 027801 (2009).
 - [17] S. M. Barnett and R. Loudon, *Phil. Trans. R. Soc. A* **368**, 927 (2010).
 - [18] S. M. Barnett, *Phys. Rev. Lett.* **104**, 070401 (2010).
 - [19] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941).
 - [20] R. A. Beth, *Phys. Rev.* **50**, 115 (1936).
 - [21] L.-S. Hou, W.-T. Ni, and Y.-C. M. Li, *Phys. Rev. Lett.* **90**, 201101 (2003).
 - [22] S. Schlamminger, K.-Y. Choi, T. A. Wagner, J. H. Gundlach, and E. G. Adelberger, *Phys. Rev. Lett.* **100**, 041101 (2008).
 - [23] I. S. Gradshteyn and I. M. Ryzhik *Table of Integrals, Series and Products* 4th ed. (Academic Press, New York, 1980).
 - [24] M. Abramowitz and I. A. Stegun *Handbook of Mathematical Functions* (Dover, New York, 1972).
 - [25] L. D. Landau and E. M. Lifshitz, *Fluid mechanics*, 2nd ed. (Pergamon Press, Oxford, 1987), Sec. 24 Problem 10.
 - [26] S. M. Spillane, T. J. Kippenberg, and K. J. Vahala, *Nature* **415**, 621 (2002).
 - [27] H. Rokshari, T. J. Kippenberg, T. Carmon, and K. J. Vahala, *Opt. Express* **13**, 5293 (2005).
 - [28] M. L. Gorodetsky, A. A. Savchenko, and V. S. Ilchenko, *Opt. Lett.* **21**, 453 (1998).
 - [29] D. W. Vernoooy, V. S. Ilchenko, H. Mabuchi, E. W. Streed, and H. J. Kimble, *Opt. Lett.* **23**, 247 (1998).
 - [30] K. J. Vahala, *Nature* **424**, 839 (2002).
 - [31] D. K. Armani, T. J. Kippenberg, S. M. Spillane, and K. J. Vahala, *Nature* **421**, 925 (2003).